

Shanghai University of Finance and Economics

2025 Entrance Examination for Advanced Interdisciplinary College

Date: September 1st, 2025

Time: 1:30 pm – 4:30 pm

There are two problems (five questions in each), and the total score is 100.

1 Pricing with Limited Knowledge of Demand

Background: In economics and operations management (运营管理), it is important to study the relationship between price and demand. Suppose a seller sells a product to a group of buyers. The seller has to decide the price of the product, which is denoted by P . Given the price P , we also define Q as the corresponding demand. It is common to use a *demand curve* (需求曲线) to represent the relationship between the price and the demand. Figure (1a) gives one such example, where the demand Q is expressed as a function of price P with $Q(P) = 3 - 2P$. Sometimes, it is also important to define the *inverse demand curve* (逆需求曲线), which is defined as the inverse function of the demand curve. Please see Figure (1b) for an example.

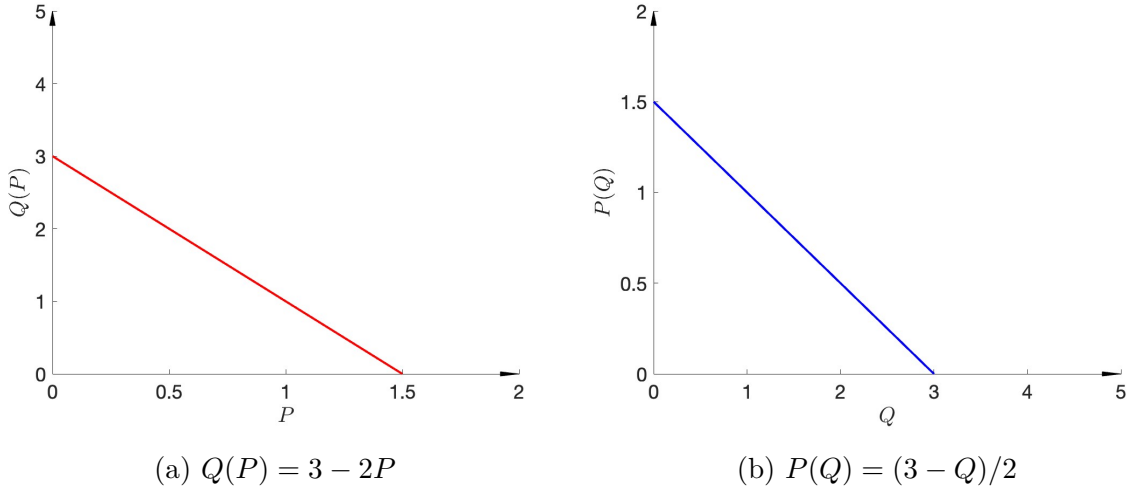


图 1: Examples of demand curve and inverse demand curve

When launching a new product to the market, it is important for the seller to determine an appropriate price. However, little is known about the demand for the new product. A natural question faced by the seller is: how to find a *simple and practical*

pricing rule for a new product when its demand information is limited? In the following, we will guide you in finding a straightforward pricing rule that performs well in various demand scenarios.

Linear demand curves are one of the most commonly used demand curves. We consider a specific form of linear demand curve, whose inverse demand function can be written as $P(Q) = P_m - bQ$. P_m can be understood as the maximum price, and $b > 0$ is the slope. We also suppose that there is a constant production cost c . Our pricing rule, denoted as P^* , is simply to set the price as the optimal price under the linear demand curve that maximizes the expected profit (期望利润).

Question 1 (5pts). Calculate P^* .

Now, we want to quantify how the pricing rule P^* performs under other demand curves. To that end, we first introduce the *monomial demand curve* (单项价格曲线). Its inverse demand function has the form $P_A(Q) = P_m - \gamma Q^n$, where $\gamma > 0$ and n is an integer with $n \geq 2$. We first study the properties of this demand function.

Convexity and Concavity are important properties of a function. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *convex* on its domain (定义域) if $f''(x) > 0$ ($f''(x)$ 表示二阶导数) for all values of x on its domain. Similarly, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be *concave* on its domain if $f''(x) < 0$ for all values of x on its domain.

Question 2 (10pts). Based on the above definitions, judge whether the function $P_A(Q)$ is convex or concave with respect to Q . Show your steps.

We now define P^{**} as the optimal price under the monomial demand curve that maximizes the expected profit.

Question 3 (10pts). Show that

$$\frac{P^{**}}{P^*} = \frac{2(nP_m + c)}{(n+1)(P_m + c)}.$$

We then want to evaluate the performance of the pricing rule P^* under the monomial demand curve. Under the monomial demand curve $P_A(Q)$, denote the expected profit achieved by P^* as Π^* and denote the expected profit achieved by P^{**} as Π^{**} .

Question 4 (10pts). Show that

$$\frac{\Pi^{**}}{\Pi^*} = \frac{2^{\frac{1}{n}+1}n}{(n+1)^{\frac{1}{n}+1}}.$$

Finally, we want to evaluate the performance of the pricing rule P^* under a general concave demand curve. Particularly, suppose now $P_A(Q)$ represents the actual demand curve, which is decreasing and concave with respect to Q and satisfies $P_A(0) \leq P_m$. Under this general $P_A(Q)$, we still denote the expected profit achieved by P^* as Π^* and denote the expected profit achieved by P^{**} , the optimal price under $P_A(Q)$, as Π^{**} .

Question 5 (15pts). Show that

$$1 \leq \frac{\Pi^{**}}{\Pi^*} \leq 2.$$

(Hint: you can directly use the result that if $P_A(Q)$ is concave, then $P^{**} \geq P^*$.)

2 Revenue Management under the MNL Model

Background: In daily life, customers make all kinds of choices frequently: for example, choosing among several brands of soda, picking among different flight tickets, or simply deciding not to buy anything. *Customer choice modeling* (消费者选择建模) is a tool that helps firms predict which option a customer will choose, and how much revenue the firm can expect. One of the most classical choice models (选择模型) in the revenue management (收益管理) literature is the *Multinomial Logit (MNL) model*. It is widely used because:

- It gives a clear formula for the probability of each choice.
- It naturally includes the possibility that a customer buys nothing (the outside option).
- It allows firms to compare different assortments (sets of products) and decide which one gives the highest expected revenue.

The key parts of the model are:

- **Assortment (选品集合) S .** The firm decides which products to offer. For example, $S = \{1, 2\}$ means the firm sells product 1 and 2, but not others.

- **Attraction value (utility weight)** $v_i > 0$. Each product i has a number v_i that shows how attractive it is to customers. A larger v_i means a higher chance of being chosen. *Example:* If $v_1 = 2$ and $v_2 = 1$, then product 1 is twice as attractive as product 2.
- **No-purchase option** $v_0 = 1$. Customers may also choose not to buy anything. We simply set its attractiveness to 1.
- **Choice probabilities (选择概率)**. If the firm offers set S , then the probability for the customer to select product i or the no-purchase option follows:

$$\Pr(i | S) = \frac{v_i}{1 + \sum_{j \in S} v_j}, \quad \Pr(0 | S) = \frac{1}{1 + \sum_{j \in S} v_j}.$$

The chance of choosing an option is proportional to its attractiveness. The no-purchase option also competes in the same way.

- **Revenue** $r_i > 0$. If product i is chosen, the firm earns revenue r_i (for example, selling price minus cost).
- **Expected revenue under S .**

$$R(S) = \sum_{i \in S} r_i \Pr(i | S) = \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i}.$$

The numerator is a revenue-weighted sum of attractiveness, while the denominator reflects the total competition, including the no-purchase option.

The following exercises will guide you through understanding some of the important properties of the MNL model. To begin with, we first consider the calculation of choice probabilities through the following example.

Question 6 (5 pts). Consider three products with attraction values $v_1 = 2$, $v_2 = 1$, $v_3 = 0.5$, and the no-purchase weight $v_0 = 1$. Let $S = \{1, 2, 3\}$. Compute

$$\Pr(1 | S), \quad \Pr(2 | S), \quad \Pr(3 | S), \quad \Pr(0 | S).$$

Which option has the highest probability of being chosen?

We then aim to prove an important property of the MNL model, known as the *Independence of Irrelevant Alternatives (IIA)* property. This property means that when

we compare two products i and j , the ratio of their choice probabilities does not change even if we add or remove other products.

Question 7 (10 pts). For the MNL model, show that for any $i, j \in S$,

$$\frac{\Pr(i \mid S)}{\Pr(j \mid S)} = \frac{v_i}{v_j}.$$

In practice, the seller has to decide the optimal set of products to offer. That is, the seller wants to find an assortment, denoted as S^* , that maximizes the expected revenue. The problem of finding this set can be described as the following optimization problem:

$$\max_S R(S) = \frac{\sum_{i \in S} r_i v_i}{1 + \sum_{i \in S} v_i}. \quad (1)$$

Problem (1) is also called the assortment optimization (选品优化) problem. To find the optimal solution to the above problem, we will subsequently derive several important structural properties of the MNL model. We first consider the fundamental question: when does adding a product to the current product increase the expected revenue?

Question 8 (15 pts) Let $S \neq \emptyset$ be an assortment with expected revenue $R(S)$. Now consider adding a new product $k \notin S$ with revenue r_k and attractiveness v_k . Show that the new expected revenue $R(S \cup \{k\})$ is greater than $R(S)$ **if and only if**

$$r_k > R(S).$$

(Hint: Compare $R(S)$ and $R(S \cup \{k\})$.)

Based on that structural property, we will first find a necessary condition for the optimal solution to problem (1).

Question 9 (10 pts). Recall that S^* is the optimal solution to problem (1). Prove:

$$\text{for all } i \in S^* : r_i \geq R(S^*), \quad \text{for all } j \notin S^* : r_j \leq R(S^*).$$

(Hint: Argue by contradiction.)

Finally, we aim to show that the optimal solution to problem (1) must follow a *revenue-ordered* (按利润排序) policy. Suppose we reindex products in decreasing order of their revenues. That is, $r_1 > r_2 > \dots > r_n$ (assuming no two products have the same revenue). Then, a revenue-ordered policy means offering products $\{1, 2, \dots, k\}$ for some

cutoff k with $1 \leq k \leq n$. For example, if $(r_1, r_2, r_3, r_4) = (10, 8, 7, 3)$, then all possible revenue-ordered assortments are $\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}$ (but not $\{1, 3\}$).

Question 10 (10 pts). Show that there must exist an optimal solution to problem (1) that is revenue-ordered.

(Hint: Using the proved result in Question 9.)